S.C. Kleene & R.E. Vesley   Foundations of Intuitionistic Mathematics   1965

$$(\beta)\qquad \forall a\gamma(a)=\begin{cases}0 \text{ if } \neg\mathrm{Seq}(a),\\ 1 \text{ if } \mathrm{Seq}(a)\ \&\ \mathrm{lh}(a)=0,\\ (\tilde{\gamma}(a))_B*2^{S+1} \text{ if } \mathrm{Seq}(a)\ \&\ \mathrm{lh}(a)\neq 0\ \&\ \sigma((\tilde{\gamma}(a))_B*2^{S+1})=0,\\ (\tilde{\gamma}(a))_B*2^{\pi((\tilde{\gamma}(a))_B)+1}\\ \qquad\qquad \text{ if } \mathrm{Seq}(a)\ \&\ \mathrm{lh}(a)\neq 0\ \&\ \sigma((\tilde{\gamma}(a))_B*2^{S+1})\neq 0\end{cases}$$

where B is $\Pi_{i<\mathrm{lh}(a)\dot{-}1}p_i^{(a)_i}$ and S is $(a)_{\mathrm{lh}(a)\dot{-}1}\dot{-}1$. If in (β) we use $\bar{\alpha}(0)$ for a (via $\forall$-elim.), the second case applies and gives $\gamma(\bar{\alpha}(0)) = \gamma(1) = 1$ (using *23.5, ˣ23.1, *B3). If in (β) we use $\bar{\alpha}(x')$ for a, then the third or fourth case applies; furthermore using *23.4, *23.2, *23.8 etc., $B=\bar{\alpha}(x)$, $S=\alpha(x)$, $\bar{\alpha}(x') = a = B\cdot p_x^{S+1} = B*2^{S+1}=\bar{\alpha}(x)*2^{\alpha(x)+1}$, so $B < a$ (using *143b, *3.10 etc.) and $(\tilde{\gamma}(a))_B = \gamma(B) = \gamma(\bar{\alpha}(x))$ (by *24.2). Now by ind., using (2) in the basis, and (α) to deal with the fourth case of (β) in the ind. step,

$$(\gamma)\qquad \sigma(\gamma(\bar{\alpha}(x)))=0.$$

Let "α_γ" abbreviate $\lambda t(\gamma(\bar{\alpha}(t')))_t\dot{-}1$. Now we deduce by induction

$$(\delta)\qquad \overline{\alpha_\gamma}(x)=\gamma(\bar{\alpha}(x)).$$

BASIS: trivial. IND. STEP. $\overline{\alpha_\gamma}(x') = \overline{\alpha_\gamma}(x)*2^{((\gamma(\bar{\alpha}(x')))_x\dot{-}1)+1}$ [*23.8, ˣ0.1] $= \gamma(\bar{\alpha}(x))*2^{((\gamma(\bar{\alpha}(x')))_x\dot{-}1)+1}$ [hyp. ind.], which (using $(\gamma(\bar{\alpha}(x))*2^{A+1})_x = (\overline{\alpha_\gamma}(x)*2^{A+1})_x$ [hyp. ind.] $= A+1$), if the third case of (β) applies to $a=\bar{\alpha}(x')$, $= \gamma(\bar{\alpha}(x))*2^{\alpha(x)+1} = \gamma(\bar{\alpha}(x'))$ {if the fourth case applies, $= \gamma(\bar{\alpha}(x))*2^{\pi(\gamma(\bar{\alpha}(x)))+1} = \gamma(\bar{\alpha}(x'))$}. — By (γ) and (δ),

$$(\varepsilon)\qquad \alpha_\gamma\in\sigma.$$

We also deduce by induction

$$(\zeta)\qquad \sigma(\bar{\alpha}(x))=0 \supset \gamma(\bar{\alpha}(x))=\bar{\alpha}(x).$$

IND. STEP. Assuming $\sigma(\bar{\alpha}(x'))=0$, the third member of (1) gives $\sigma(\bar{\alpha}(x))=0$, so by hyp. ind. $\gamma(\bar{\alpha}(x))=\bar{\alpha}(x)$, and the third case of (β) applies. — By (δ), (ζ), *23.2 and *6.3, $\sigma(\bar{\alpha}(x'))=0 \supset \alpha_\gamma(x)=\alpha(x)$, whence

$$(\eta)\qquad \alpha\in\sigma \supset \alpha_\gamma=\alpha.$$

II. Assume also the remaining hyps. **(3)–(6)** of *26.4a. We shall apply ˣ26.3a with $R(\gamma(a))$, $A(\gamma(a))$ as the $R(a)$, $A(a)$. If we can then verify the four hyps. of ˣ26.3a, the concl. of *26.4a will follow using $\gamma(1)=1$ (in I). We get the first hyp. by (γ) with (3) (using *23.6 to put $a=\bar{\alpha}(x)$ preparatory to $\exists$-elims.). For the second, by (ε) and (4) $\exists xR(\overline{\alpha_\gamma}(x))$, whence by (δ) $\exists xR(\gamma(\bar{\alpha}(x)))$. We get the third (putting $a=\bar{\alpha}(x)$) by (γ) with (5). For the fourth, assume $\mathrm{Seq}(a)\ \&\ \forall sA(\gamma(a*2^{s+1}))$. By (γ) with *23.6, $\sigma(\gamma(a))=0$. Put $x=\mathrm{lh}(a)$. Assuming $\sigma(\gamma(a)*2^{s+1})=0$, and using *22.8, *22.5, *23.6 to put $a*2^{s+1}=\bar{\alpha}(y)$ (then $y=x'$ [*22.8, *20.3, *23.5], $a\cdot p_x^{s+1} = a*2^{s+1}$ [ˣ21.1 etc.] $= \bar{\alpha}(x') = \bar{\alpha}(x)\cdot p_x^{\alpha(x)+1}$ [*23.8], so $s=\alpha(x)$ [*19.11, *22.2, *19.9, *6.3] and $a=\bar{\alpha}(x)$ [*133]), the third case of (β) applies to $\bar{\alpha}(x')$ and gives $\gamma(\bar{\alpha}(x'))=\gamma(a)*2^{s+1}$, so $\forall sA(\gamma(a*2^{s+1}))$ gives $A(\gamma(a)*2^{s+1})$; thus $\forall s\{\sigma(\gamma(a)*2^{s+1})=0 \supset A(\gamma(a)*2^{s+1})\}$. By (6), $A(\gamma(a))$.

6.10. From his bar theorem Brouwer inferred his "fan theorem" (implicit in **1923a** p. 4 (II); **1924** Theorem 2; **1927** Theorem 2; **1954** § 5). A "finite set" or "finitary spread", most recently called a *fan*, is a spread in which each choice must be from a finite collection of numbers. Say e.g. that, for $t = 0, 1, 2, \ldots$, the number $\alpha(t)$ must be chosen from among $0, 1, \ldots, \beta(\bar{\alpha}(t))$; i.e. $(t)\alpha(t)\leq\beta(\bar{\alpha}(t))$. We shall here be considering only the choice sequences underlying a fan, which constitute a fan by taking for the correlation law ρ the trivial correlation $\rho(\bar{\alpha}(x')) = \alpha(x)$. According to one version of the fan theorem (classically true), if, for all choice sequences α restricted to this fan (determined by β), $(Ex)R(\bar{\alpha}(x))$, then there is a finite upper bound z to the least x's for which $R(\bar{\alpha}(x))$. In this "pure" version, symbolized by *26.6a (or *26.6b–*26.6d), we can prove the fan theorem from the bar theorem with no further postulate. Another version *27.7 (classically false), favored by Brouwer, will follow from this by the new intuitionistic postulate ˣ27.1 of § 7. A classical contrapositive of the present version is König's lemma **1926**, which we shall give in Remark 9.11.

First, we give a proof of the present version of the fan theorem informally. Consider any sequence number a belonging to the given fan, i.e. representing a finite choice sequence belonging to that fan; by the *subfan issuing from a* we mean the fan of those choice sequences α by which a can be extended in the given fan, i.e. such that, for each x, the sequence number $a*\bar{\alpha}(x)$ represents a finite choice sequence belonging to that fan. We apply Brouwer's **1927** Footnote 7 in 6.5 above, but considering only sequence numbers not past secured belonging to the given fan: "for every s ($s = 0, 1, 2, \ldots$)" becomes "for every $s \leq \beta(a)$". We use the corresponding form of induction to prove as follows that, under the hyp. of the fan theorem for the given fan and the given predicate R, the conclusion of the fan theorem   Pf fro
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holds for the subfan issuing from any sequence number $a = \delta(y)$ securable but not past secured (in the given fan with respect to the given R) and the predicate $\lambda w\, R(a*w)$. The subfan issuing from a sequence number a such that R(a) has 0 as a z for the fan theorem. Consider a sequence number a whose securability follows from that of all $a*2^{s+1}$ for $s \leq \beta(a)$; by the hyp. ind., for each $s \leq \beta(a)$ the subfan issuing from $a*2^{s+1}$ has a z, call it z_s, for the fan theorem. So the subfan issuing from a has $1+\max(z_0, \ldots, z_{\beta(a)})$ as a z for the fan theorem. This completes the induction. But under the hyp. of the fan theorem, 1 is securable but not past secured. So the conclusion of the fan theorem holds for the subfan issuing from 1 and the predicate $\lambda w\, R(1*w)$, i.e. for the given fan and R.

This is easily pictured geometrically. Our fan is represented by a tree in which from each vertex, occupied by the sequence number a, finitely many arrows (namely $\beta(a)+1$ of them) lead to vertices, occupied by $a*2^{0+1}, \ldots, a*2^{\beta(a)+1}$. This is illustrated by Figure 1 in 6.5 for the case $(a)[\beta(a)=2]$ (the *binary fan*), where now we are not to imagine arrows for $s > 1$. Again consider a predicate $R(a)$; and suppose that, for each α, we underline the first $\bar{\alpha}(x)$ (if any) for which $R(\bar{\alpha}(x))$. Figure 1 illustrates a case in which $(\alpha)(Ex)R(\bar{\alpha}(x))$. To simplify terminology, let us suppress in each branch all vertices to the right of an underlined $\bar{\alpha}(x)$; so in Figure 1 only the part of the tree printed in bold face remains. The hypothesis of the fan theorem then says that all paths are finite. The conclusion says that there is a finite upper bound to their lengths. The proof is by induction, corresponding to the inductive definition of the class of the securable (but not past secured) sequence numbers a (6.5, but now in the fan rather than in the universal spread). The induction proposition is that there is a finite upper bound to the lengths of paths in the subtree issuing from a. As basis of the induction, this upper bound is 1 (the z is 0) for a at the end of any branch. As induction step, in proceeding leftward from all $a*2^{s+1}$ ($s = 0, 1$ in Figure 1) to a, we graft finitely many subtrees (2 in our Figure 1) with respective finite upper bounds onto a to obtain a subtree with upper bound the maximum of the respective upper bounds increased by one.

In formalizing this proof, we first prove a lemma *26.5, in which b, s, z, w are any distinct number variables, and B(s, z) is any formula not containing b, w free in which w is free for z.

*26.5. $\forall s \forall z \forall w[B(s, z) \;\&\; w \geq z \supset B(s, w)]$
$\vdash \forall s_{s \leq b} \exists z B(s, z) \supset \exists z \forall s_{s \leq b} B(s, z)$.

PROOF. We assume **(a)** $\forall s \forall z \forall w[B(s, z) \;\&\; w \geq z \supset B(s, w)]$, and deduce the rest by ind. on b. IND. STEP. Assume $\forall s_{s \leq b'} \exists z B(s, z)$, whence $\exists z B(b', z)$ and $\forall s_{s \leq b} \exists z B(s, z)$. By hyp. ind., $\exists z \forall s_{s \leq b} B(s, z)$. Assume for $\exists$-elim., $B(b', z_1)$ and $\forall s_{s \leq b} B(s, z_2)$. Using (a) and *8.4, $B(b', \max(z_1, z_2))$ and $\forall s_{s \leq b} B(s, \max(z_1, z_2))$, whence $\forall s_{s \leq b'} B(s, \max(z_1, z_2))$, whence $\exists z \forall s_{s \leq b'} B(s, z)$.

*26.6a. $\vdash \forall a[\mathrm{Seq}(a) \supset R(a) \vee \neg R(a)] \;\&\; \forall \alpha_{B(\alpha)} \exists x R(\bar{\alpha}(x)) \supset$
$\exists z \forall \alpha_{B(\alpha)} \exists x_{x \leq z} R(\bar{\alpha}(x))$

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where $B(\alpha)$ is $\forall t\, \alpha(t) \leq \beta(\bar{\alpha}(t))$.

*26.6d. $\vdash \forall \alpha_{B(\alpha)} \exists x[R(\bar{\alpha}(x)) \;\&\; \forall y_{y<x} \neg R(\bar{\alpha}(y))] \supset$
$\exists z \forall \alpha_{B(\alpha)} \exists x_{x \leq z}[R(\bar{\alpha}(x)) \;\&\; \forall y_{y<x} \neg R(\bar{\alpha}(y))]$.

PROOF OF *26.6a. I. $B(\alpha)$ does indeed restrict α to a non-empty spread. For, we can introduce a function variable σ so that the following formula (a) holds. Specifically, using #22, #D, #E, etc., the right member of (a) is equivalent to $p(a)=0$ for some term $p(a)$ (with $\vdash p(a) \leq 1$). Using Lemma 5.3 (a), assume preparatory to $\exists$-elim. $\forall a[\sigma(a)=p(a)]$. Thence

(a) $\forall a[\sigma(a)=0 \sim \mathrm{Seq}(a) \;\&\; \forall t_{t<\mathrm{lh}(a)} (a)_t \dot{-} 1 \leq \beta(\Pi_{i<t} p_i^{(a)_i})]$.

(The following also proves *26.6a′ in which the "Seq(a)" of *26.6a is replaced by the right side of (a).) By *23.2, *23.4, *23.5 and *6.3,

(b) $\sigma(\bar{\alpha}(x))=0 \sim \forall t_{t<x} \alpha(t) \leq \beta(\bar{\alpha}(t))$.

Thence

(c) $B(\alpha) \sim \alpha \epsilon \sigma$.

Furthermore, the first two hyps **(1)** $\mathrm{Spr}(\sigma)$ (using 0 for the s in the second member) and **(2)** $\sigma(1)=0$ of *26.4a now hold.

II. We shall apply *26.4a with the present σ and R taking A(a) as follows.

A(a): $\exists z \forall \alpha[\forall t\, \alpha(t) \leq \beta(a*\bar{\alpha}(t)) \supset \exists x_{x \leq z} R(a*\bar{\alpha}(x))]$.

With this A(a), the concl. $\exists z \forall \alpha_{B(\alpha)} \exists x_{x \leq z} R(\bar{\alpha}(x))$ of *26.6a will follow from A(1) by *22.7. So it will suffice, assuming the two hyps. of *26.6a, to deduce the other four hyps. **(3)–(6)** of *26.4a. The next three (3)–(5) we quickly obtain (with 0 for the z in (5)). To deduce (6), assume **(d)** $\sigma(a)=0$ and **(e)** $\forall s\{\sigma(a*2^{s+1})=0 \supset A(a*2^{s+1})\}$; we must deduce $A(a)$. Using (d), (a) and *23.6, we can put (for $\exists$-elims.) **(f)** $a=\bar\delta(y)$. Then by (d) and (b): **(g)** $\forall t_{t<y}\delta(t)\leq\beta(\bar\delta(t))$. We shall deduce $A(\bar\delta(y))$, i.e. $\exists z\forall\alpha[\forall t\alpha(t)\leq\beta(\bar\delta(y)*\bar\alpha(t)) \supset \exists x_{x\leq z}R(\bar\delta(y)*\bar\alpha(x))]$.

A. Assume **(h)** $s\leq\beta(\bar\delta(y))$. Using *23.6 with *22.5 and *22.8, we can put (for $\exists$-elims.) $\bar\delta(y)*2^{s+1}=\overline{\delta'}(u)$, whereupon (by *23.5, *22.8, *20.3) $u=y'$, so $\bar\delta(y)*2^{s+1}=\overline{\delta'}(y')$. By *23.2, ×21.1, *19.11 with *23.3 and *19.9, $\delta'(y)=s$; with *23.2 and *19.10, $t<y \supset \delta'(t)=\delta(t)$; hence by *B19 with ×23.1, $t\leq y \supset \overline{\delta'}(t)=\bar\delta(t)$. Now (g) and (h) give $\forall t_{t<y'}\delta'(t)\leq\beta(\overline{\delta'}(t))$, whence by (b) $\sigma(\overline{\delta'}(y'))=0$, whence $\sigma(\bar\delta(y)*2^{s+1})=0$, whence by (e) and (f) $A(\bar\delta(y)*2^{s+1})$, i.e. $\exists z\forall\alpha[\forall t\alpha(t)\leq\beta((\bar\delta(y)*2^{s+1})*\bar\alpha(t)) \supset \exists x_{x\leq z}R((\bar\delta(y)*2^{s+1})*\bar\alpha(x))]$; call this formula $\exists zB(s, z)$. By the $\exists u$- and $\exists\delta'$-elim., and $\supset$- and $\forall$-introd., $\forall s_{s\leq\beta(\bar\delta(y))}\exists zB(s, z)$. But $B(s, z)$ has the property expressed by the assumption formula of *26.5. Hence $\exists z\forall s_{s\leq\beta(\bar\delta(y))}B(s, z)$.

B. Assume (for $\exists$-elim.) $\forall s_{s\leq\beta(\bar\delta(y))}B(s, z)$, and $\forall t\alpha(t)\leq\beta(\bar\delta(y)*\bar\alpha(t))$. Now $\alpha(0)\leq\beta(\bar\delta(y))$, and so $B(\alpha(0), z)$, i.e. $\forall\alpha'[\forall t\alpha'(t)\leq\beta((\bar\delta(y)*2^{\alpha(0)+1})*\overline{\alpha'}(t)) \supset \exists x_{x\leq z}R((\bar\delta(y)*2^{\alpha(0)+1})*\overline{\alpha'}(x))]$. Let "$\alpha'$" abbreviate $\lambda t\alpha(t')$. Then $\alpha'(t) = \alpha(t')$ [×0.1] $\leq \beta(\bar\delta(y)*\bar\alpha(t')) = \beta(\bar\delta(y)*(\bar\alpha(1)*\overline{\alpha'}(t)))$ [*23.7] $= \beta((\bar\delta(y)*2^{\alpha(0)+1})*\overline{\alpha'}(t))$ [*22.9]; so $\forall t\alpha'(t)\leq\beta((\bar\delta(y)*2^{\alpha(0)+1})*\overline{\alpha'}(t))$. So from $B(\alpha(0), z)$, $\exists x_{x\leq z}R((\bar\delta(y)*2^{\alpha(0)+1})*\overline{\alpha'}(x))$. Assume $x\leq z$ & $R((\bar\delta(y)*2^{\alpha(0)+1})*\overline{\alpha'}(x))$. Thence $x'\leq z'$ & $R(\bar\delta(y)*\bar\alpha(x'))$, whence $\exists x_{x\leq z'}R(\bar\delta(y)*\bar\alpha(x))$. By the $\exists x$-elim., $\supset$-, $\forall$- and $\exists z$-introd., and the $\exists z$-elim., $\exists z\forall\alpha[\forall t\alpha(t)\leq\beta(\bar\delta(y)*\bar\alpha(t)) \supset \exists x_{x\leq z}R(\bar\delta(y)*\bar\alpha(x))]$.

More generally, the choices permitted for $\alpha(t)$ in a fan need not be a non-empty initial segment of the natural numbers. The choice law is then a function σ satisfying the first two hypotheses of:

*26.7a. $\vdash \mathrm{Spr}(\sigma)$ & $\forall a[\sigma(a)=0 \supset \exists b\forall s\{\sigma(a*2^{s+1})=0 \supset s\leq b\}]$ & $\forall a[\sigma(a)=0 \supset R(a) \vee \neg R(a)]$ & $\forall\alpha_{\alpha\in\sigma}\exists xR(\bar\alpha(x))$ $\supset \exists z\forall\alpha_{\alpha\in\sigma}\exists x_{x\leq z}R(\bar\alpha(x))$.

PROOF. CASE 1: $\sigma(1)\neq 0$. Then $\neg\alpha\in\sigma$. Use *10a. (The fan is empty and the theorem holds vacuously.)

CASE 2: $\sigma(1)=0$. Assume the four hyps. **(1′)–(4′)** of *26.7a.

I. We have the first two hyps. (1) and (2) of *26.4a, so we can introduce π and γ as in I of the proof there and (α)–(η) will hold.

II. Using $\sigma(a)=0 \vee \sigma(a)\neq 0$ and (2′), $\forall a\exists b[\sigma(a)=0 \supset \forall s\{\sigma(a*2^{s+1})=0 \supset s\leq b\}]$. Applying *2.2, we may assume for $\exists$-elim.

$$(\theta) \qquad \forall a[\sigma(a)=0 \supset \forall s\{\sigma(a*2^{s+1})=0 \supset s\leq\beta(a)\}].$$

III. We shall apply *26.6a for the β of (θ) (entering into $B(\alpha)$) with $R(\gamma(a))$ (for the γ of I) as the $R(a)$.

A. First we verify that the concl. of *26.7a will then follow from the concl. $\exists z\forall\alpha_{B(\alpha)}\exists x_{x\leq z}R(\gamma(\bar\alpha(x)))$ of *26.6a. Assume for $\exists$-elim. **(ι)** $\forall\alpha_{B(\alpha)}\exists x_{x\leq z}R(\gamma(\bar\alpha(x)))$. Assume **$(\varkappa)$** $\alpha\in\sigma$, whence $\sigma(\bar\alpha(t))=0$ and $\sigma(\bar\alpha(t'))=0$. But $\bar\alpha(t')=\bar\alpha(t)*2^{\alpha(t)+1}$. Applying (θ), $\alpha(t)\leq\beta(\bar\alpha(t))$; and by $\forall$-introd., $B(\alpha)$. Hence by (ι), $\exists x_{x\leq z}R(\gamma(\bar\alpha(x)))$. Omitting $\exists x$ for $\exists$-elim., we have $x\leq z$ and $R(\gamma(\bar\alpha(x)))$; by $(\varkappa)$, $\sigma(\bar\alpha(x))=0$. So by (ζ), $R(\bar\alpha(x))$. By &-, $\exists$- and $\supset$-introd., (completing) the $\exists x$-elim., $\forall$- and $\exists$-introd., and the $\exists z$-elim., $\exists z\forall\alpha_{\alpha\in\sigma}\exists x_{x\leq z}R(\bar\alpha(x))$.

B. It remains for us to verify the two hyps. of *26.6a. For the first, assume $\mathrm{Seq}(a)$, and put $a=\bar\alpha(x)$. By (γ) $\sigma(\gamma(\bar\alpha(x)))=0$, so by (3′) $R(\gamma(a)) \vee \neg R(\gamma(a))$. For the second, by (ε) and (4′) $\exists xR(\overline{\alpha_\gamma}(x))$, whence by (δ) $\exists xR(\gamma(\bar\alpha(x)))$, whence by *11 $B(\alpha) \supset \exists xR(\gamma(\bar\alpha(x)))$, whence by $\forall$-introd. $\forall\alpha_{B(\alpha)}\exists xR(\gamma(\bar\alpha(x)))$.

6.11. We now formalize the induction principle in the bar theorem for inferring a property A of any barred sequence number w (cf. 6.6 ¶s 2, 3). This gives us an axiom schema modelled directly on Axiom Schema 13 for ordinary induction. In this the implication of an inductive by an explicit sense of securability, barredness, etc. (i.e. the reversal of direction, 6.5 ¶ 1), which is the kernel of the bar theorem, enters thus: the conclusion of the induction that each barred sequence number w has the property A is formulated using the explicit sense of 'barred'. The intuitionistic restriction on the predicate R with respect to which numbers w are barred we give as a preliminary hypothesis, in two forms. (For a classical result, corresponding to *26.1, we may omit the first hyp. of ×26.8a.)

×26.8a. $\forall a[\mathrm{Seq}(a) \supset R(a) \vee \neg R(a)]$ & $\forall a[\mathrm{Seq}(a)$ & $R(a) \supset A(a)]$ & $\forall a[\mathrm{Seq}(a)$ & $\forall sA(a*2^{s+1}) \supset A(a)] \supset \{\mathrm{Seq}(w)$ & $\forall\alpha\exists xR(w*\bar\alpha(x)) \supset A(w)\}$.

×26.8c. $\forall\alpha\forall x\forall y[R(\bar\alpha(x))$ & $R(\bar\alpha(y)) \supset x=y]$ & $\forall a[\mathrm{Seq}(a)$ & $R(a) \supset A(a)]$ & $\forall a[\mathrm{Seq}(a)$ & $\forall sA(a*2^{s+1}) \supset A(a)] \supset \{\mathrm{Seq}(w)$ & $\forall\alpha\exists xR(w*\bar\alpha(x)) \supset A(w)\}$.